A canonical family of multiple orthogonal polynomials for Nikishin systems

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Abstract

For any pair of compact intervals Δ_1 and Δ_2 of the real line such that $\Delta_1 \cap \Delta_2 = \emptyset$ we obtain two pairs of absolutely continuous **probability** measures (μ_1, μ_2) and (τ_1, τ_2) supported on Δ_1 and Δ_2, respectively, such that

· for appropriate constants C_1 and C_2, (μ_1, μ_2) is the Nikishin system generated by $(\mu_1, C_1\tau_1)$ and (τ_1, τ_2) the Nikishin system generated by $(\tau_1, C_2\mu_1)$,

· the polynomials of multiple orthogonality with respect to the Nikishin system (μ_1, μ_2) and indices $\{\ldots, (n, n), (n + 1, n), \ldots\}$ satisfy a recurrence relations with *constant* coefficients of period 2,

· $1/\widehat{\mu}_1(z)$ and $1/\widehat{\mu}_2(z)$ are the functions which describe the ratio asymptotics of multiple orthogonal polynomials with respect to an arbitrary Nikishin system $\mathcal{N}(\sigma_1, \sigma_2)$ verifying $\mathrm{supp}(\sigma_i) = \Delta_i$, and $\sigma_i' > 0, i = 1, 2,$ almost everywhere on Δ_i. Analogously, $1/\widehat{\tau}_1(z)$ and $1/\widehat{\tau}_2(z)$ give the ratio asymptotics for $\mathcal{N}(\sigma_2, \sigma_1)$.

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1 Introduction

Let $\Sigma = (\sigma_1, \sigma_2)$ be a system of positive Borel measures such that, for each $k = 1, 2$, σ_k is supported on a compact interval $\Delta_k = [a_k, b_k] \subset \mathbb{R}$ (which does not reduce to a single point) and $\Delta_1 \cap \Delta_2 = \emptyset$. Let $\mathcal{N}(\Sigma) = (s_1, s_2)$ be the Nikishin system generated by Σ. That is,

$$ds_1(x) = d\sigma_1(x), \qquad ds_2(x) = \widehat{\sigma}_2(x) d\sigma_1(x).$$

For any measure μ,

$$\widehat{\mu}(z) = \int \frac{d\mu(x)}{z - x}.$$

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The general definition of a Nikishin system was given in [14] where the author called them MT systems. Such systems have attracted great attention in connection with the extension of the asymptotic theory of orthogonal polynomials and the convergence of simultaneous Hermite-Padé approximation.

For each multi-index $\mathbf{n} = (n_1, n_2) \in \mathbf{I}$, where

$$\mathbf{I} = \{(0,0), (1,0), (1,1) \ldots, (2,1), \ldots\},$$

we consider the *monic* multiple orthogonal polynomial $Q_{\mathbf{n}}$ satisfying

$$\int x^\nu Q_{\mathbf{n}}(x) ds_k(x) = 0, \quad 0 \le \nu \le n_k - 1, \quad k = 1, 2.$$

In [14] it was shown that any multi-index $(n_1, n_2) \in \mathbf{I}$ is normal. This means that $\deg Q_{\mathbf{n}} = n_1 + n_2$ and is uniquely determined by the orthogonality conditions and has degree $|\mathbf{n}| = n_1 + n_2$. (Regarding normality, for more general classes of multi-indices and Nikishin systems, see also [5], [6], and [7]). Since for any $\mathbf{n} \in \mathbf{I}$ there exist unique $k \in \mathbb{Z}_+$ and $i \in \{0, 1\}$ such that $|\mathbf{n}| = 2k + i$, for convenience we write $Q_{2k+i} = Q_{\mathbf{n}}$.

Consider the 3-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{2} \mathcal{R}_k},$$

formed by the consecutive "glued" sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \quad \mathcal{R}_1 := \overline{\mathbb{C}} \setminus (\Delta_1 \cup \Delta_2), \quad \mathcal{R}_2 := \overline{\mathbb{C}} \setminus \Delta_2,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Let F_i, $i = 1, 2$, be the single valued rational function on $\mathcal{R}$ whose divisor consists of a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(i)}$ with the normalization

$$F_i(x) = x + \mathcal{O}(1), \qquad x \to \infty^{(0)}, \tag{1}$$

and let $\{F_{i,k}\}_{k=0}^2$ be the different branches of F_i corresponding to the different sheets $\mathcal{R}_k$ of $\mathcal{R}$. When $\operatorname{supp}(\sigma_k) = \Delta_k$ and $\sigma_k' > 0$ a.e. on Δ_k, $k = 1, 2$, in [2, Theorem 1.2] it was proved that

$$\lim_{n \to \infty} \frac{Q_{2n+i}(z)}{Q_{2n+i-1}(z)} = F_{i,0}(z), \quad i = 1, 2, \tag{2}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_1$. This result on ratio asymptotics extends E.A. Rakhmanov's theorem given for standard orthogonal polynomials on the real line.

Remarks.

1. The limits $F_{i,0}(z)$ are expressed in [2, Theorem 1.2] in a different way, but the equivalence between both expressions is an immediate consequence of Liouville's theorem and Cardano's formulas.

2. Ratio asymptotics of multiple orthogonal polynomials for Nikishin systems also hold when each σ_k is supported on $\Delta_k \cup e_k$, where e_k is a denumerable set without accumulation points in $\mathbb{R} \setminus \Delta_k$, and $\sigma_k' > 0$ a.e.. This is proved in [13, Theorem 4.1].

It is well known and easy to verify that the polynomials $Q_n(x)$ satisfy the recurrence relations

$$\begin{aligned} &Q_{-2}(x) = Q_{-1}(x) = 0, \quad Q_0(x) = 1 \\ &xQ_n(x) = Q_{n+1}(x) + \alpha_n^* Q_n(x) + \beta_n^* Q_{n-1}(x) + \gamma_n^* Q_{n-2}(x), \quad n = 0, 1, \ldots \end{aligned} \tag{3}$$

for certain real constants $\alpha_n^*, \beta_n^*, \gamma_n^*$. Moreover, as proved in [3, Theorem 1.2], when the polynomials have ratio asymptotics as expressed in (2), the coefficients of the recurrence have limits with period 2. More precisely, there exist real constants $\alpha_i, \beta_i, \gamma_i$ such that

$$\lim_{n\to\infty} \alpha_{2n+i}^* = \alpha_i, \quad \lim_{n\to\infty} \beta_{2n+i}^* = \beta_i, \quad \lim_{n\to\infty} \gamma_{2n+i}^* = \gamma_i, \quad i = 0, 1.$$

Remark. When $\Delta_1 = [a, 0]$, $\Delta_2 = [0, 1]$ and there is ratio asymptotics, the values of the constants $\alpha_i, \beta_i, \gamma_i$, $i = 0, 1$, are given in [3, Section 4.5]. Notice that in this case $\Delta_1 \cap \Delta_2 = \{0\} \neq \emptyset$ and is not considered in the present paper.

Here, we prove that

$$\left(\frac{1}{F_{1,0}(z)}, \frac{1}{F_{2,0}(z)}\right) = (\widehat{\mu}_1(z), \widehat{\mu}_2(z)), \qquad \frac{\alpha_1 - \alpha_0}{\gamma_1}\left(\frac{F_{2,2}(z)}{F_{1,2}(z)}, F_{2,2}(z)\right) = (\widehat{\tau}_1(z), \widehat{\tau}_2(z)),$$

where μ_k are probabilities supported on Δ_1, τ_k are probabilities supported on Δ_2, and

$$d\mu_2(x) = \frac{\gamma_0}{(\alpha_1 - \alpha_0)^2}\widehat{\tau}_1(x) d\mu_1(x), \qquad d\tau_2(x) = \frac{-\gamma_0}{(\alpha_1 - \alpha_0)^2}\widehat{\mu}_1(x) d\tau_1(x).$$

Moreover, the sequence of polynomials $(P_n)_{n=0}^{\infty}$, defined by the recurrences

$$\begin{gathered} xP_{2n+i}(x) = P_{2n+i+1}(x) + \alpha_k P_{2n+i}(x) + \beta_k P_{2n+i-1}(x) + \gamma_k P_{2n+i-2}(x), \\ i = 0, 1, \quad n = 0, 1, \ldots \end{gathered} \tag{4}$$

with initial conditions $P_{-2} = P_{-1} = 0$, $P_0 = 1$, corresponds to the sequence of multiple orthogonality with respect to the Nikishin system $(\mu_1, \mu_2) = \mathcal{N}(\mu_1, \frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\tau_1)$. If the starting point was a Nikishin system on (Δ_2, Δ_1), then $\frac{1}{\widehat{\tau}_1(z)}$ and $\frac{1}{\widehat{\tau}_2(z)}$ would give the ratio asymptotics for the corresponding multiple orthogonal polynomials.

Since the polynomials (4) are defined by recurrences with (periodic) constant coefficients they are the analogues of the second kind Chebyshev polynomials in standard orthogonality. Therefore, we call them Chebyshev-Nikishin multiple orthogonal polynomials, and Chebyshev-Nikishin measures the corresponding orthogonality measures.

In Section 2, we obtain the algebraic equations satisfied by the functions F_k in terms of the constants $\alpha_i, \beta_i, \gamma_i$ as well as some relations between these constants. In Section 3, the measures are described and it is proved that they form a Nikishin system. Finally, in Section 4, we prove the orthogonality conditions for the sequence of polynomials $(P_n)_{n=0}^{\infty}$.

2 Algebraic equations

First of all, let us find the equations satisfied by the algebraic functions F_1 and F_2 in terms of the limits of the recurrence coefficients.

Theorem 2.1. *i) The coefficients of the recurrence relation* (4) *verify*

$$\beta_0 = \beta_1 := \beta, \quad \gamma_1 = \gamma_0 + \beta(\alpha_1 - \alpha_0). \tag{5}$$

ii) The functions F_1, F_2, given in (1) *satisfy $F_1 - F_2 = \alpha_1 - \alpha_0$.*

iii) The algebraic equations for F_i are

$$\begin{aligned} &F_1^3 - (x + \alpha_1 - 2\alpha_0)F_1^2 + \{(\alpha_1 - \alpha_0)(x - \alpha_0) + \beta\}F_1 + \gamma_0 = 0 \\ &F_2^3 - (x + \alpha_0 - 2\alpha_1)F_2^2 + \{(\alpha_0 - \alpha_1)(x - \alpha_1) + \beta\}F_2 + \gamma_1 = 0. \end{aligned}$$

Proof. Since $F_1(z) - F_2(z)$ is a bounded function on the compact Riemann surface $\mathcal{R}$ (it is sufficient to look at the divisor of both functions), by Liouville's theorem, this function is constant. Let $C \neq 0$ be the constant such that $F_1(z) - F_2(z) \equiv C$. Then, $C = -F_2(\infty^{(1)}) = F_1(\infty^{(2)})$ since $F_k(\infty^{(k)}) = 0$.

From the recurrence relation (3) we have

$$x \frac{Q_{2n+k}(x)}{Q_{2(n-1)+k}(x)} =$$

$$\frac{Q_{2n+k+1}(x)}{Q_{2(n-1)+k}(x)} + \alpha^*_{2n+k} \frac{Q_{2n+k}(x)}{Q_{2(n-1)+k}(x)} + \beta^*_{2n+k} \frac{Q_{2n+k-1}(x)}{Q_{2(n-1)+k}(x)} + \gamma^*_{2n+k}.$$

Taking limits as $n \to \infty$, $k = 0, 1$, using (2) we obtain algebraic expressions on the sheet $\mathcal{R}^{(0)}$ which can be extended to the whole Riemann surface which allow us to write

$$\begin{aligned} z\,Z &= F_1(z)\,Z + \alpha_0\,Z + \beta_0\,F_1(z) + \gamma_0, \\ z\,Z &= F_2(z)\,Z + \alpha_1\,Z + \beta_1\,F_2(z) + \gamma_1, \quad z \in \mathcal{R}, \end{aligned} \tag{6}$$

where $Z = F_1(z)F_2(z)$. Then

$$\lim_{z\to\infty^{(k)}} (F_1(z)\, Z + \alpha_0\, Z + \beta_0\, F_1(z) + \gamma_0) = \lim_{z\to\infty^{(k)}} (F_2(z)\, Z + \alpha_1\, Z + \beta_1\, F_2(z) + \gamma_1)$$

and for $k = 1$ and $k = 2$ we get

$$\begin{aligned} \gamma_0 &= \beta_1 F_2(\infty^{(1)}) + \gamma_1 \\ \beta_0 F_1(\infty^{(2)}) + \gamma_0 &= \gamma_1. \end{aligned} \tag{7}$$

Then $\gamma_0 = -\beta_1 C + \beta_0 C + \gamma_0$ and $\beta_0 = \beta_1 := \beta$ follows. Using that $F_1(z) - F_2(z) \equiv C$, deleting one equation in (6) from the other, we deduce that

$$0 \equiv (C + (\alpha_0 - \alpha_1))Z + \beta C + \gamma_0 - \gamma_1.$$

Evaluating at $\infty^{(1))}$ (or $\infty^{(2))}$), it follows that $\beta C + \gamma_0 - \gamma_1 = 0$. Therefore, $F_1(z) - F_2(z) = \alpha_1 - \alpha_0 = C$ and $\gamma_1 = \gamma_0 + \beta(\alpha_1 - \alpha_0)$. Finally, the equations of $F_1(z)$ and $F_2(z)$ can be obtained after writing F_2 in terms of F_1 or F_1 in terms of F_2 in (6). $\square$

Remark. In [11, Theorem 3.1] a system of equations is given which allows to derive the algebraic equations satisfied by F_1, F_2, in terms of the endpoints of the intervals Δ_1, Δ_2. Combining those results with ours you can obtain the values $\gamma_1, \gamma_2, \beta, \alpha_1, \alpha_2$ in terms of the endpoints of the intervals.

Now, consider the functions of second type

$$\Phi_n(x) = \int \frac{Q_n(t)}{x-t}\, d\sigma_1(t), \qquad \Psi_n(x) = \int \frac{\Phi_n(t)}{x-t}\, d\sigma_2(t).$$

In [9, Proposition 1] it is proved that for the multi-index $(n_1, n_2) \in \mathbf{I}$ and $n = n_1 + n_2$,

$$\int x^\nu \Phi_n(x)\, d\sigma_2(x) = 0,\ \ \nu = 0, 1, \ldots, n_2 - 1.$$

Consequently, $\Phi_n(x)$ has at least n_2 zeros on Δ_2. Moreover, it is shown in [9, Proposition 3] that $\Phi_n(x)$ has exactly n_2 zeros in $\mathbb{C} \setminus \Delta_1$, they are all simple, and lie in the interior of Δ_2. Let $Q_{n,2}(x)$ be the monic polynomial defined by the zeros of $\Phi_n(x)$. They also prove in [9, Proposition 2] that $Q_n(x)$ and $Q_{n,2}(x)$ satisfy full orthogonality relations with respect to certain varying measure. More precisely,

$$\int x^\nu Q_n(x) \frac{d\sigma_1(x)}{Q_{n,2}(x)} = 0, \quad 0 \le \nu \le n-1,$$

$$\int x^\nu Q_{n,2}(x) \int \frac{Q_n^2(t)}{x-t} \frac{d\sigma_1(t)}{Q_{n,2}(t)} \frac{d\sigma_2(x)}{Q_n(x)} = 0, \quad 0 \le \nu \le n_2 - 1.$$

When $\operatorname{supp}(\sigma_k) = \Delta_k$ and $\sigma_k' > 0$ a.e. on Δ_k, $k = 1, 2$, in [2, Theorem 4.1] it is proved that the polynomials $Q_{n,2}$ also have ratio asymptotics; that is, there exist functions $R_1(z)$ and $R_2(z)$ such that

$$\lim_{n\to\infty} \frac{Q_{2n+i,2}(z)}{Q_{2n+i-1,2}(z)} = R_i(z), \quad i = 0, 1,$$

uniformly on compact sets of $\mathbb{C} \setminus \Delta_2$. Moreover, if we denote $\widehat{Q}_n(x) = \kappa_n Q_n(x)$, $\kappa_n > 0$, the orthonormal polynomial with respect to $\frac{d\sigma_1(x)}{|Q_{n,2}(x)|}$ and $\widehat{Q}_{n,2}(x) = \kappa_{n,2} Q_{n,2}, \kappa_{n,2} > 0$, the orthonormal polynomial with respect to the varying measure $\int \frac{\widehat{Q}_n^2(t)}{|x-t|} \frac{d\sigma_1(t)}{|Q_{n,2}(t)|} \frac{d\sigma_2(x)}{|Q_n(x)|}$, in [2, Corollary 4.1] the ratio asymptotics of $\widehat{Q}_n(x)$ and $\widehat{Q}_{n,2}(x)$ is deduced. In particular, this gives that the following limits exist and are different from zero

$$\lim_{n\to\infty} \frac{\kappa_{2n+i}}{\kappa_{2n+i-1}} := \kappa_i, \qquad \lim_{n\to\infty} \frac{\kappa_{2n+i,2}}{\kappa_{2n+i-1,2}} := \kappa_{i,2}, \quad i = 0, 1.$$

Then, using the orthogonality relations satisfied by the polynomials, we have

$$\frac{\Phi_{2n+i}(x)}{\Phi_{2n+i-1}(x)} = \frac{\int \frac{Q_{2n+i}(t)}{x-t} d\sigma_1(t)}{\int \frac{Q_{2n+i-1}(t)}{x-t} d\sigma_1(t)} = \frac{Q_{2n+i,2}(x)}{Q_{2n+i-1,2}(x)} \frac{Q_{2n+i-1}(x)}{Q_{2n+i}(x)} \frac{\int \frac{Q_{2n+i}^2(t)}{x-t} \frac{d\sigma_1(t)}{Q_{2n+i,2}(t)}}{\int \frac{Q_{2n+i-1}^2(t)}{x-t} \frac{d\sigma_1(t)}{Q_{2n+i-1,2}(t)}} \tag{8}$$

$$= \frac{Q_{2n+i,2}(x)}{Q_{2n+i-1,2}(x)} \frac{Q_{2n+i-1}(x)}{Q_{2n+i}(x)} \frac{\kappa_{2n+i-1}^2}{\kappa_{2n+i}^2} \frac{\int \frac{\widehat{Q}_{2n+i}^2(t)}{|x-t|} \frac{d\sigma_1(t)}{|Q_{2n+i,2}(t)|}}{\int \frac{\widehat{Q}_{2n+i-1}^2(t)}{|x-t|} \frac{d\sigma_1(t)}{|Q_{2n+i-1,2}(t)|}} c_i(\Delta_1, \Delta_2)$$

where $c_i(\Delta_1, \Delta_2)$ is 1 or -1 depending on i and on the relative position of Δ_1 and Δ_2. In [10, Corollary 3], it is proved that the measures $\widehat{Q}_n^2(t) \frac{d\sigma_1(t)}{|Q_{n,2}(t)|}$ are converge weakly to the Chebyshev measure of the interval Δ_1. Consequently, using (8) and the results mentioned above, we find that there exist functions $A_i(x)$ such that

$$\lim_{n\to\infty} \frac{\Phi_{2n+i}(x)}{\Phi_{2n+i-1}(x)} = A_i(x), \ i = 0, 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus (\Delta_1 \cup \Delta_2)$.

Analogously, one proves that there exist functions $B_i(x)$ such that

$$\lim_{n\to\infty} \frac{\Psi_{2n+i}(x)}{\Psi_{2n+i-1}(x)} = B_i(x), \ i = 0, 1,$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_2$.

On the other hand, from the orthogonality relations satisfied by Q_n with respect to σ_1 and of $\Phi_n(x)$ with respect to σ_2 it follows that

$$x\Phi_n(x) = \int \frac{xQ_n(t)}{x-t} d\sigma_1(t) = \int \frac{tQ_n(t)}{x-t} d\sigma_1(t), \ n \geq 1,$$

and

$$x\Psi_n(x) = \int \frac{x\Phi_n(t)}{x-t} d\sigma_2(t) = \int \frac{t\Phi_n(t)}{x-t} d\sigma_2(t), \ n \geq 2.$$

Consequently, the functions $\Phi_n, n \geq 1$ and $\Psi_n, n \geq 2$, satisfy the same recurrence relations as the polynomials $Q_n, n \geq 0$.

Theorem 2.2. *The functions of second kind satisfy*

$$\lim_{n\to\infty} \frac{\Phi_{2n+i}(z)}{\Phi_{2n+i-1}(z)} = F_{i,1}(z), \qquad \lim_{n\to\infty} \frac{\Psi_{2n+i}(z)}{\Psi_{2n+i-1}(z)} = F_{i,2}(x), \quad i = 1, 2,$$

and the convergence is uniform on compact subsets of $\mathbb{C}\backslash(\Delta_1 \cup \Delta_2))$ *and* $\mathbb{C}\backslash\Delta_2$, *respectively.*

Proof. Since $\Phi_n(x), n \geq 1$, satisfy the same recurrence relations as $Q_n(x)$, we may deduce as in Theorem 2.1 the equations (6) for A_0 and A_1, obtaining

$$\frac{\beta A_1 + \gamma_0}{x - \alpha_0 - A_1} = A_0 A_1 = \frac{\beta A_0 + \gamma_1}{x - \alpha_1 - A_0}. \tag{9}$$

Therefore,

$$(\beta A_1 + \gamma_0)(x - \alpha_1 - A_0) = (\beta A_0 + \gamma_1)(x - \alpha_0 - A_1)$$

or, equivalently,

$$(\beta(x - \alpha_1) + \gamma_1)A_1 - (\beta(x - \alpha_0) + \gamma_0)A_0 = \gamma_1(x - \alpha_0) - \gamma_0(x - \alpha_1).$$

Taking into account that $\gamma_1 = \gamma_0 + \beta(\alpha_1 - \alpha_0)$, the last equality leads to

$$(\beta(x - \alpha_0) + \gamma_0)(A_1 - A_0) = (\alpha_1 - \alpha_0)(\beta(x - \alpha_0) + \gamma_0).$$

Should $\beta = \gamma_0 = 0$, on account of (6), we would have $F_1 = x - \alpha_0$ and $F_2 = x - \alpha_1$ which is impossible. Therefore, $A_1 - A_0 = \alpha_1 - \alpha_0$. Substituting A_0 in terms of A_1 in the first equality of (9) and A_1 in terms of A_0 in the second equality, we see that A_1 and A_0 satisfy the same algebraic equations as F_1 and F_2, respectively. From (8), it is easy to deduce that $A_0(\infty) \in \mathbb{C} \setminus \{0\}$ and $A_1(\infty) = 0$. Therefore, $A_1 = F_{1,1}$ and $A_0 = F_{2,1}$. Analogously, one proves that $B_1 = F_{1,2}$ and $B_0 = F_{2,2}$. □

Dividing $xQ_n(x)$ by $Q_n(x)$ in the recurrence relation and taking limits, one sees that

$$F_{1,0}(z) = z - \alpha_0 + \mathcal{O}(1/z), \qquad F_{2,0}(z) = z - \alpha_1 + \mathcal{O}(1/z) \qquad z \to \infty.$$

Using that $F_1 - F_2 = \alpha_1 - \alpha_0$, $F_{1,1}(\infty) = F_{2,2}(\infty) = 0$, and that from the algebraic equations $\prod_{k=0}^{2} F_{1,k}(z) = -\gamma_0$, $\prod_{k=0}^{2} F_{2,k}(z) = -\gamma_1$, one immediately obtains that

$$\begin{aligned} F_{1,1}(z) &= \frac{-\gamma_0}{\alpha_1 - \alpha_0}\frac{1}{z} + \mathcal{O}(1/z^2), \quad & F_{1,2}(z) &= \alpha_1 - \alpha_0 + \mathcal{O}(1/z), \\ F_{2,1}(z) &= \alpha_0 - \alpha_1 + \mathcal{O}(1/z), & F_{2,2}(z) &= \frac{\gamma_1}{\alpha_1 - \alpha_0}\frac{1}{z} + \mathcal{O}(1/z^2). \end{aligned} \tag{10}$$

3 Chebyshev-Nikishin measures

From the proof of [2, Theorem 2.1] we know that the analytic functions $Q_n(x)$ and $\Phi_n(x)$ satisfy

$$\begin{aligned} &Q'_{n+1}(x)Q_n(x) - Q_{n+1}(x)Q'_n(x) \neq 0, \; x \in \Delta_1, \\ &\Phi'_{n+1}(x)\Phi_n(x) - \Phi_{n+1}(x)\Phi'_n(x) \neq 0, \; x \in \Delta_2. \end{aligned}$$

This allowed the authors to show that the zeros of the polynomials Q_n and $Q_{n,2}$ interlace those of Q_{n+1} and $Q_{n+1,2}$. An immediate consequence is that in the decomposition in simple fractions of $\frac{Q_n(x)}{Q_{n+1}(x)}$ and $\frac{Q_{n+1,2}(x)}{Q_{n,2}(x)}$ all the residues have the same sign (recall that the zeros of Q_n and $Q_{n,2}$ are simple). Therefore, since we are considering monic polynomials, and there is ratio aymptotics, there exist probability measures $\mu_1,\ \mu_2$ supported on Δ_1, and $\tau_1,\ \tau_2$ supported on Δ_2, such that

$$\begin{aligned} &\lim_{n\to\infty} \frac{Q_{2n}(z)}{Q_{2n+1}(z)} = \widehat{\mu}_1(z), &&\lim_{n\to\infty} \frac{Q_{2n-1}(z)}{Q_{2n}(z)} = \widehat{\mu}_2(z), \\ &\lim_{n\to\infty} \frac{Q_{2n+1,2}(z)}{Q_{2n,2}(z)} = 1 + C\widehat{\tau}_1(z), &&\lim_{n\to\infty} \frac{Q_{2n-1,2}(z)}{Q_{2n,2}(z)} = \widehat{\tau}_2(z), \end{aligned}$$

for some real constant C.

Lemma 3.1. *The constant C in the definition of $\widehat{\tau}_1(z)$ is equal to $\frac{-\gamma_1}{(\alpha_1-\alpha_0)^2}$ and the measures satisfy the relations*

$$\begin{aligned} &i) \;\; \widehat{\mu}_2(z) - \widehat{\mu}_1(z) = (\alpha_1 - \alpha_0)\, \widehat{\mu}_1(z)\widehat{\mu}_2(z), \\ &ii) \;\; \widehat{\mu}_2(z) - \widehat{\tau}_2(z) = \frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\, \widehat{\mu}_1(z)\widehat{\tau}_1(z), \\ &iii) \;\; \widehat{\tau}_2(z) - \widehat{\tau}_1(z) = \frac{\gamma_1}{(\alpha_1-\alpha_0)^2}\, \widehat{\tau}_1(z)\widehat{\tau}_2(z). \end{aligned}$$

Proof. From (8), (10), Theorem 2.2, and the fact that the measures are probabilistic, it is clear that

$$\begin{aligned} &F_{1,0}(z) = \frac{1}{\widehat{\mu}_1(z)}, \quad F_{1,1}(z) = \frac{\gamma_0}{\alpha_0 - \alpha_1}(1 + C\widehat{\tau}_1(z))\widehat{\mu}_1(z), \quad F_{1,2}(z) = \frac{\alpha_1 - \alpha_0}{1 + C\widehat{\tau}_1(z)}, \\ &F_{2,0}(z) = \frac{1}{\widehat{\mu}_2(z)}, \quad F_{2,1}(z) = (\alpha_0 - \alpha_1)\frac{\widehat{\mu}_2(z)}{\widehat{\tau}_2(z)}, \qquad\qquad F_{2,2}(z) = \frac{\gamma_1}{\alpha_1 - \alpha_0}\widehat{\tau}_2(z). \end{aligned}$$

Using these relations, the equality $F_{1,0}(z) - F_{2,0}(z) = \alpha_1 - \alpha_0$ immediately gives $i)$. The expression $F_{1,2}(z) - F_{2,2}(z) = \alpha_1 - \alpha_0$ means that

$$\frac{\alpha_1 - \alpha_0}{1 + C\widehat{\tau}_1(z)} - \frac{\gamma_1}{\alpha_1 - \alpha_0}\widehat{\tau}_2(z) = \alpha_1 - \alpha_0$$

which is equivalent to $-\gamma_1(1+C\widehat{\tau}_1(z))\widehat{\tau}_2(z)=(\alpha_1-\alpha_0)^2C\widehat{\tau}_1(z)$. Due to the behavior at ∞, this implies that $C=\frac{-\gamma_1}{(\alpha_1-\alpha_0)^2}$. Consequently, $(1+C\widehat{\tau}_1(z))\widehat{\tau}_2(z)=\widehat{\tau}_1(z)$ which is *iii*).

Now, from $F_{1,1}(z)-F_{2,1}(z)=\alpha_1-\alpha_0$, we deduce that

$$\frac{\gamma_0}{\alpha_0-\alpha_1}(1+C\widehat{\tau}_1(z))\widehat{\mu}_1(z)-(\alpha_0-\alpha_1)\frac{\widehat{\mu}_2(z)}{\widehat{\tau}_2(z)}=\alpha_1-\alpha_0$$

and using that $(1+C\widehat{\tau}_1(z))\widehat{\tau}_2(z)=\widehat{\tau}_1(z)$ we obtain *ii*). □

From the existing relations between the Cauchy transform of the measures μ_k, τ_k, and the algebraic functions F_1 and F_2, it is clear that these measures are absolutely continuous. We denote by μ_k' and τ_k' their Radon-Nikodym derivatives. We express by $\widehat{\mu}_k^{\pm}$ and $\widehat{\tau}_k^{\pm}$ the boundary values of these functions on $\Delta_1^{\pm}$ and $\Delta_2^{\pm}$ respectively.

The measures μ_1 and τ_1 define the Nikishin systems (μ_1,μ_2), (τ_1,τ_2). In the same way that (μ_1,μ_2) gives the ratio asymptotics of polynomials defined by a Nikishin system on Δ_1 in which the second generating measure is supported on Δ_2, (τ_1,τ_2) gives the ratio asymptotics of polynomials defined by a Nikishin system on Δ_2 for which the second measure of the generating system is supported on Δ_1. This is the next result.

Theorem 3.1. *The measures μ_k and τ_k, $k=1,2$, satisfy the following relations*

$$d\mu_2(x)=\frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\,\widehat{\tau}_1(x)d\mu_1(x),\qquad d\tau_2(x)=\frac{-\gamma_0}{(\alpha_1-\alpha_0)^2}\widehat{\mu}_1(x)d\tau_1(x).$$

Moreover, $\widehat{\tau}_1(z)$ and $\widehat{\tau}_2(z)$ are the functions which describe the asymptotic behavior of orthogonal polynomials of a Nikishin system on Δ_2 where the second generating measure is taken on Δ_1.

Proof. Since $\widehat{\mu}_1(z)\widehat{\tau}_1(z)$ vanishes at infinity, we can write

$$\widehat{\mu}_1(z)\widehat{\tau}_1(z)=\frac{1}{2\pi i}\int_\Gamma\frac{\widehat{\mu}_1(\xi)\widehat{\tau}_1(\xi)}{z-\xi}d\xi$$

where Γ is a positively oriented closed simple curve with Δ_1 and Δ_2 in its interior and z in the exterior of Γ. For $k=1,2$, let Γ_k be any positively oriented closed simple curve surrounding Δ_k which leaves Δ_j, $k\neq j$, in the exterior. Then

$$\widehat{\mu}_1(z)\widehat{\tau}_1(z)=\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\widehat{\mu}_1(\xi)\widehat{\tau}_1(\xi)}{z-\xi}\,d\xi+\frac{1}{2\pi i}\int_{\Gamma_2}\frac{\widehat{\mu}_1(\xi)\widehat{\tau}_1(\xi)}{z-\xi}\,d\xi.$$

Taking limits when Γ_1 and Γ_2 shrink to Δ_1 and Δ_2 respectively, using the Sokhotski formula, we obtain

$$\widehat{\mu}_1(z)\widehat{\tau}_1(z)=\frac{1}{2\pi i}\int_{\Delta_1}\frac{(\widehat{\mu}_1^-(x)-\widehat{\mu}_1^+(x))\widehat{\tau}_1(x)}{z-x}\,dx+\frac{1}{2\pi i}\int_{\Delta_2}\frac{(\widehat{\tau}_1^-(x)-\widehat{\tau}_1^+(x))\widehat{\mu}_1(x)}{z-x}\,dx$$

$$= \int_{\Delta_1} \frac{\mu_1'(x)\widehat{\tau}_1(x)}{z-x}\,dx + \int_{\Delta_2} \frac{\tau_1'(x)\widehat{\mu}_1(x)}{z-x}\,dx$$

From $ii)$ of Lemma 3.1, it follows that

$$\int_{\Delta_1} \left(\frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\mu_1'(x)\widehat{\tau}_1(x) - \mu_2'(x) \right) \frac{dx}{z-x}$$

$$= -\int_{\Delta_2} \left(\frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\widehat{\mu}_1(x)\tau_1'(x) + \tau_2'(x) \right) \frac{dx}{z-x}.$$

Then, both integrals represent entire functions. Since they take the value zero at infinity, by Liouville's theorem, both integrals are identically equal to zero and we conclude that

$$d\mu_2(x) = \frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\widehat{\tau}_1(x)d\mu_1(x), \qquad d\tau_2(x) = -\frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\widehat{\mu}_1(x)d\tau_1(x).$$

On the other hand, by Sokhotski's formula

$$\mu_2'(x) = \frac{1}{2\pi i}\left(\frac{1}{F_{2,0}^-(x)} - \frac{1}{F_{2,0}^+(x)} \right) = \frac{1}{2\pi i}\left(\frac{F_{2,0}^+(x) - F_{2,0}^-(x)}{F_{2,0}^-(x)F_{2,0}^+(x)} \right) = \frac{1}{2\pi i}\frac{F_{2,0}^+(x) - F_{2,1}^+(x)}{F_{2,0}^+(x)F_{2,1}^+(x)}$$

$$= \frac{1}{2\pi i}\frac{(F_{1,0}^+(x) - F_{1,1}^+(x))F_{2,2}(x)}{-\gamma_1}, \quad x \in \Delta_1.$$

Analogously,

$$\mu_1'(x) = \frac{1}{2\pi i}\left(\frac{1}{F_{1,0}^-(x)} - \frac{1}{F_{1,0}^+(x)} \right) = \frac{1}{2\pi i}\frac{(F_{1,0}^+(x) - F_{1,1}^+(x))F_{1,2}(x)}{-\gamma_0}, \quad x \in \Delta_1.$$

Hence $\frac{\mu_1'(x)}{\mu_2'(x)} = \frac{\gamma_1 F_{1,2}(x)}{\gamma_0 F_{2,2}(x)}$, $x \in \Delta_1$. But $\mu_2'(x) = \frac{\gamma_0}{(\alpha_1-\alpha_0)^2}\widehat{\tau}_1(x)\mu_1'(x)$ and, consequently,

$$\frac{1}{\widehat{\tau}_1(z)} = \frac{\gamma_1}{(\alpha_1-\alpha_0)^2}\frac{F_{1,2}(z)}{F_{2,2}(z)}, \quad z \in \mathbb{C}\setminus\Delta_2.$$

Notice that $\frac{F_1}{F_2} = 1 + \frac{\alpha_1-\alpha_0}{F_2}$ lives on $\mathcal{R}$ and has for divisor a simple pole at $\infty^{(2)}$ and a simple zero at $\infty^{(1)}$. Except for a constant factor, these are the characteristics which define one of the functions which describes the ratio asymptotics of Nikishin polynomials on Δ_2 with the second generating measure on Δ_1. So this function must be $\dfrac{1}{\widehat{\tau}_1(z)}$. We have analogous conclusions for $\widehat{\tau}_2$ since

$$\frac{1}{\widehat{\tau}_2(z)} = \frac{\gamma_1}{\alpha_1-\alpha_0}\frac{1}{F_{2,2}(z)}, \quad z \in \mathbb{C}\setminus\Delta_2.$$

With this we conclude the proof. □

4 Chebyshev-Nikishin polynomials

With the recurrence (3) of the Nikishin polyomials $(Q_n)_{n=0}^{\infty}$ we can associate the matrices

$$A_n(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha^*_{2n} & -\beta^*_{2n} & x-\gamma^*_{2n} \end{pmatrix}, \qquad B_n(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha^*_{2n+1} & -\beta^*_{2n+1} & x-\gamma^*_{2n+1} \end{pmatrix},$$

and we have

$$\begin{pmatrix} Q_{2n}(x) \\ Q_{2n+1}(x) \\ Q_{2n+2}(x) \end{pmatrix} = B_{n-1}(x)A_{n-1}(x) \begin{pmatrix} Q_{2(n-1)}(x) \\ Q_{2(n-1)+1}(x) \\ Q_{2(n-1)+2}(x) \end{pmatrix}.$$

Let $A(x) = \lim_{n\to\infty} A_n(x)$ and $B(x) = \lim_{n\to\infty} B_n(x)$. Then

$$B(x)A(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma_1 & -\beta & x-\alpha_1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma_0 & -\beta & x-\alpha_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ -\gamma_0 & -\beta & x-\alpha_0 \\ -\gamma_0(x-\alpha_1) & -\gamma_1-\beta(x-\alpha_1) & -\beta+(x-\alpha_1)(x-\alpha_0) \end{pmatrix}.$$

From the vector version of the Poincaré-Perron theorem (see, for example, [4, pag. 1750]), $\lim_{n\to\infty} \dfrac{Q_{2n+2}(x)}{Q_{2n}(z)} = \lim_{n\to\infty} \dfrac{Q_{2n+1}(z)}{Q_{2n-1}(z)} = F_{1,0}(z)F_{2,0}(z)$ is some eigenvalue of the matrix $B(x)A(x)$. As a consequence, $Z := F_1F_2$ satisfies the equation

$$0 = \det(B(z)A(z) - Z\ I)$$

$$= Z^3 - \{(z-\alpha_0)(z-\alpha_1) - 2\beta\}Z^2 + \{\beta^2 + \gamma_1(z-\alpha_0) + \gamma_0(x-\alpha_1)\}Z - \gamma_0\gamma_1.$$

Let $(P_n)_{n=0}^{\infty}$ and $(P^*_n)_{n=0}^{\infty}$ be the sequences of monic polynomials defined by the recurrences

$$\begin{aligned} xP_{2n}(x) &= P_{2n+1}(x) + \alpha_0 P_{2n}(x) + \beta P_{2n-1}(x) + \gamma_0 P_{2n-2}(x), \quad n=0,1,\ldots \\ xP_{2n+1}(x) &= P_{2n+2}(x) + \alpha_1 P_{2n+1}(x) + \beta P_{2n}(x) + \gamma_1 P_{2n-1}(x), \quad n=0,1,\ldots \end{aligned}$$

$$\begin{aligned} xP^*_{2n}(x) &= P^*_{2n+1}(x) + \alpha_1 P^*_{2n}(x) + \beta P^*_{2n-1}(x) + \gamma_1 P^*_{2n-2}(x), \quad n=0,1,\ldots \\ xP^*_{2n+1}(x) &= P^*_{2n+2}(x) + \alpha_0 P^*_{2n+1}(x) + \beta P^*_{2n}(x) + \gamma_0 P^*_{2n-1}(x), \quad n=0,1,\ldots \end{aligned}$$

with initial conditions $P_0(x) = P_0^*(x) = 1,\ P_{-2}(x) = P_{-2}^*(x) = P_{-1}(x) = P_{-1}^*(x) = 0.$ Then

$$\begin{pmatrix} P_{2n}(x) \\ P_{2n+1}(x) \\ P_{2n+2}(x) \end{pmatrix} = (B(x)A(x))^{n+1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} P_{2n}^*(x) \\ P_{2n+1}^*(x) \\ P_{2n+2}^*(x) \end{pmatrix} = (A(x)B(x))^{n+1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Using the relation $\frac{\beta F_1+\gamma_0}{x-F_1-\alpha_0} = Z = \frac{\beta F_2+\gamma_1}{x-F_2-\alpha_1}$ deduced from (6) one can prove that $(1, F_{1,k}, Z_k)$ is an eigenvector associated to the eigenvalue Z_k for $B(x)A(x)$ and $k = 0, 1, 2$. Taking into account that $A(x)B(x)$ has the same expression as $B(x)A(x)$ but interchanging the indices 0 and 1 in the entries of the matrix, it follows that $(1, F_{2,k}, Z_k),\ k = 0, 1, 2,$ are eigenvectors for $A(x)B(x)$. Then we have

$$B(x)A(x) = P(x)D(x)P^{-1}(x), \qquad A(x)B(x) = Q(x)D(x)Q^{-1}(x)$$

with

$$P(x) = \begin{pmatrix} 1 & 1 & 1 \\ F_{1,0} & F_{1,1} & F_{1,2} \\ Z_0 & Z_1 & Z_2 \end{pmatrix}, \ Q(x) = \begin{pmatrix} 1 & 1 & 1 \\ F_{2,0} & F_{2,1} & F_{2,2} \\ Z_0 & Z_1 & Z_2 \end{pmatrix}, \ D(x) = \begin{pmatrix} Z_0 & 0 & 0 \\ 0 & Z_1 & 0 \\ 0 & 0 & Z_2 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} P_{2n} \\ P_{2n+1} \\ P_{2n+2} \end{pmatrix} = PD^{n+1}P^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{1,3}Z_0^{n+1} + a_{2,3}Z_1^{n+1} + a_{3,3}Z_2^{n+1} \\ a_{1,3}F_0Z_0^{n+1} + a_{2,3}F_1Z_1^{n+1} + a_{3,3}F_2Z_2^{n+1} \\ a_{1,3}Z_0^{n+2} + a_{2,3}Z_1^{n+2} + a_{3,3}Z_2^{n+2} \end{pmatrix},$$

where $a_{i,j}$ for $i, j \in \{1, 2, 3\}$ are the entries of P^{-1}.

The matrices P and Q have the same determinant because

$$\det P =$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ F_{1,0} & F_{1,1} & F_{1,2} \\ Z_0 & Z_1 & Z_2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ F_{1,0} & F_{1,1} & F_{1,2} \\ F_{1,0}^2 & F_{1,1}^2 & F_{1,2}^2 \end{vmatrix} = \prod_{2\geq j>k\geq 0} (F_{1,j} - F_{1,k}) = \prod_{2\geq j>k\geq 0} (F_{2,j} - F_{2,k})$$

$$= \det Q.$$

Moreover,

$$P^{-1}\det P = \begin{pmatrix} \begin{vmatrix} F_{1,1} & Z_1 \\ F_{1,2} & Z_2 \end{vmatrix} & -\begin{vmatrix} 1 & Z_1 \\ 1 & Z_2 \end{vmatrix} & \begin{vmatrix} 1 & F_{1,1} \\ 1 & F_{1,2} \end{vmatrix} \\ -\begin{vmatrix} F_{1,0} & Z_0 \\ F_{1,2} & Z_2 \end{vmatrix} & \begin{vmatrix} 1 & Z_0 \\ 1 & Z_2 \end{vmatrix} & -\begin{vmatrix} 1 & F_{1,0} \\ 1 & F_{1,2} \end{vmatrix} \\ \begin{vmatrix} F_{1,0} & Z_0 \\ F_{1,1} & Z_1 \end{vmatrix} & -\begin{vmatrix} 1 & Z_0 \\ 1 & Z_1 \end{vmatrix} & \begin{vmatrix} 1 & F_{1,0} \\ 1 & F_{1,1} \end{vmatrix} \end{pmatrix} \tag{11}$$

and it is clear that, if $a_{k,j}$ are the coefficients of P^{-1} and $b_{k,j}$ are the coefficients of Q^{-1}, then $a_{k,j} = b_{k,j}$ for $j = 2, 3$ and for all k, and $b_{k,1} = a_{k,1} - (\alpha_1 - \alpha_0)a_{k,2},\ k = 1, 2, 3$. As a consequence, we have the following representation for the polynomials,

$$\begin{gathered} P_{2n}(x) = a_{1,3}Z_0^{n+1} + a_{2,3}Z_1^{n+1} + a_{3,3}Z_2^{n+1}, \qquad P^*_{2n}(x) = P_{2n}(x), \\ P_{2n+1}(x) = a_{1,3}F_{1,0}\, Z_0^{n+1} + a_{2,3}F_{1,1}\, Z_1^{n+1} + a_{3,3}F_{1,2}\, Z_2^{n+1}, \\ P^*_{2n+1}(x) = a_{1,3}F_{2,0}\, Z_0^{n+1} + a_{2,3}F_{2,1}\, Z_1^{n+1} + a_{3,3}F_{2,2}\, Z_2^{n+1}, \\ P^*_{2n+1}(x) = P_{2n+1}(x) - (\alpha_1 - \alpha_0)P_{2n}(x). \end{gathered} \tag{12}$$

Remark. Since $|Z_0| > |Z_1|$ and $|Z_0| > |Z_2|$ in a neighborhood of infinity, formulas (12) mean that the polynomials P_{2n} and P_{2n+1} have strong asymptotics and they behave like $Z_0^n = (F_{1,0}F_{2,0})^n$ in $\mathbb{C} \setminus \Delta_1$. This coincides with the general result on strong asymptotics given in [1] for multiple orthogonal polynomials associated with Nikishin systems defined by absolutely continuous measures in the Szegő class.

Theorem 4.1. *i)* $(P_n)_{n=0}^{\infty}$ *is the sequence of monic multiple orthogonal polynomials associated with the Nikishin system* (μ_1, μ_2) *and the multi-indices of the form* (n, n) *and* $(n + 1, n)$.

ii) $(P^*_n)_{n=0}^{\infty}$ *is the sequence of monic multiple orthogonal polynomials associated with the Nikishin system* (μ_1, μ_2) *and the multi-indices of the form* (n, n) *and* $(n, n + 1)$.

Proof. Let $Z = F_1F_2$. Recall that Z has a double pole at $\infty^{(0)}$ and simple zeros at $\infty^{(1)}$ and $\infty^{(2)}$. From (12), taking into account that $\widehat{\mu}_1(z) = \frac{1}{F_{1,0}(z)}$, $\widehat{\mu}_2(z) = \frac{1}{F_{2,0}(z)}$, that the determinant of P has a double pole at infinity, and the expressions of $a_{k,3}(z)$ given in (11), we can write the following identities:

$$\widehat{\mu}_1(z)P_{2n}(z) = a_{1,3}(z)\frac{Z_0^{n+1}}{F_{1,0}} + a_{2,3}(z)\frac{Z_1^{n+1}}{F_{1,0}} + a_{3,3}(z)\frac{Z_2^{n+1}}{F_{1,0}}$$

$$= a_{1,3}(z)F_{2,0}Z_0^n + a_{2,3}(z)F_{2,1}Z_1^n + a_{3,3}(z)F_{2,2}Z_2^n + a_{2,3}(z)\left(\frac{Z_1}{F_{1,0}} - F_{2,1}\right)Z_1^n +$$

$$+a_{3,3}(z)\left(\frac{Z_2}{F_{1,0}} - F_{2,2}\right)Z_2^n = P_{2n-1}^*(z) + \mathcal{O}(1/z^{n+1}), \quad z \to \infty,$$

$$\widehat{\mu}_2(z)P_{2n}^*(z) = a_{1,3}(z)F_{1,0}Z_0^n + a_{2,3}(z)\frac{Z_1^{n+1}}{F_{2,0}} + a_{3,3}(z)\frac{Z_2^{n+1}}{F_{2,0}}$$

$$= P_{2n-1}(z) + \mathcal{O}(1/z^{n+1}), \quad z \to \infty,$$

$$\widehat{\mu}_1(z)P_{2n+1}(z) = a_{1,3}(z)Z_0^n + a_{2,3}(z)\frac{F_{1,1}Z_1^{n+1}}{F_{1,0}} + a_{3,3}(z)\frac{F_{1,2}Z_2^{n+1}}{F_{1,0}}$$

$$= P_{2n}(z) + a_{2,3}(z)\left(\frac{F_{1,1}}{F_{1,0}} - 1\right)Z_1^{n+1} + a_{3,3}(z)\left(\frac{F_{1,2}}{F_{1,0}} - 1\right)Z_2^{n+1}$$

$$= P_{2n}(z) + \mathcal{O}(1/z^{n+2}), \quad z \to \infty,$$

and

$$\widehat{\mu}_2(z)P_{2n+1}^*(z) = a_{1,3}(z)Z_0^{n+1} + a_{2,3}(z)\frac{F_{2,1}Z_1^{n+1}}{F_{2,0}} + a_{3,3}(z)\frac{F_{2,2}Z_2^{n+1}}{F_{2,0}}$$

$$= P_{2n}(z) + \mathcal{O}(1/z^{n+2}), \quad z \to \infty, .$$

If Γ is a simple closed Jordan curve that surrounds Δ_1, using Cauchy's integral formula and Fubini's theorem, from these relations it follows that

$$0 = \frac{1}{2\pi i}\int_\Gamma z^\nu P_{2n}(z)\widehat{\mu}_1(z)dz = \int x^\nu P_{2n}(x)d\mu_1(x), \qquad \nu = 0, \ldots, n-1,$$

$$0 = \frac{1}{2\pi i}\int_\Gamma z^\nu P_{2n}^*(z)\widehat{\mu}_2(z)dz = \int x^\nu P_{2n}^*(x)d\mu_2(x), \qquad \nu = 0, \ldots, n-1,$$

$$0 = \frac{1}{2\pi i}\int_\Gamma z^\nu P_{2n+1}(z)\widehat{\mu}_1(z)dz = \int x^\nu P_{2n+1}(x)d\mu_1(x), \qquad \nu = 0, \ldots, n,$$

and

$$0 = \frac{1}{2\pi i}\int_\Gamma z^\nu P_{2n+1}^*(z)\widehat{\mu}_2(z)dz = \int x^\nu P_{2n+1}^*(x)d\mu_2(x), \qquad \nu = 0, \ldots, n,$$

Taking into consideration that $P_{2n} = P_{2n}^*$ and $P_{2n+1}^*(x) = P_{2n+1}(x) - (\alpha_1 - \alpha_0)P_{2n}(x)$ (see (12)), the rest of the orthogonality relations immediately follow and we are done. □

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